

Observing the temperature of the Big Bang through large scale structure

Pedro G. Ferreira¹ and João Magueijo^{2,3,4}

¹*Astrophysics, University of Oxford, Oxford, OX1 3RH*

²*Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo N2L 2Y5, Canada*

³*Canadian Institute for Theoretical Astrophysics, 60 St George St, Toronto M5S 3H8, Canada*

⁴*Theoretical Physics, Imperial College, London, SW7 2BZ*

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It is widely accepted that the Universe underwent a period of thermal equilibrium at very early times. One expects a residue of this primordial state to be imprinted on the large scale structure of space time. In this paper we study the morphology of this thermal residue in a universe whose early dynamics is governed by a scalar field. We calculate the amplitude of fluctuations on large scales and compare it to the imprint of vacuum fluctuations. We then use the observed power spectrum of fluctuations on the cosmic microwave background to place a constraint on the temperature of the Universe before and during inflation. We also present an alternative scenario where the fluctuations are predominantly thermal and near scale-invariant.

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A cornerstone of modern cosmology is that the universe underwent a sustained period of thermal equilibrium at early times. Two of the key predictions of the big bang cosmology, the spectrum of the cosmic microwave background and the abundance of light elements, hinge on the existence of this primordial hot phase [1]. A key characteristic of systems in thermal equilibrium is the presence of fluctuations. These are, to some extent, uniquely defined and can be derived from the microphysical properties of the system [2]. Hence we expect to be able to characterize the fluctuations of the energy density of the early universe which in turn lead to irregularities in the fabric in space time. These should be reflected in the distribution of large scale structure, the propagation of light rays and other such cosmological observables.

There have been a number of attempts at pinning down the fine details of the thermal fluctuations in the early universe. Under standard assumptions it can be shown that thermal models are observationally unsound. A generic $n_s = 4$ prediction for the spectral index follows, unless there is a phase transition, in which case $n_s = 0$. This can be bypassed, and a more congenial $n_s \approx 1$ be predicted, by considering non-standard assumptions: e.g. by considering a gas of strings at the Hagedorn phase [3], or by invoking an early holographic phase in loop quantum cosmology [4], followed by a phase transition. One can also appeal to the technicalities of loop quantum cosmology [5] or postulate a mildly sub-extensive contribution to the energy density [7]. All these scenarios require speculative new physics.

In this paper, we revisit this issue by focusing on what has become a standard and fruitful model of the universe: a perturbed homogeneous and isotropic spacetime whose dynamics is driven by a scalar field. Without loss of generality, we will restrict ourselves to a scalar field with an exponential potential but will allow both positive and negative kinetic energies [9]. If the field rolls sufficiently slowly away from the origin, we have power law, accelerated expansion. If the field rolls sufficiently quickly, the

energy density in the scalar field will mimic the behaviour of an assortment of cosmological fluids (such as radiation or dust). If the kinetic energy of the scalar field is negative, we obtain “phantom”-like behaviour: the effective equation of state $w \equiv P/\rho$ (where ρ and P are the energy density and pressure in the scalar field) is such that $w < -1$. Such a setup allows us to analytically calculate the amplitude and spectrum of thermal fluctuations including gravitational backreaction. In this paper we will focus on universes that underwent superluminal expansion.

Let us briefly revisit the model. We will consider a potential for the scalar field of the form: $V(\phi) = M_{Pl}^4 \exp(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{Pl}})$ where M_{Pl} is the reduced Planck mass. The evolution of the scalar field is given by $\phi = \sqrt{2p} M_{Pl} \ln(M_{Pl} t / \sqrt{p(3p-1)})$ and the Friedman equations lead to a simple solution of the form $a \propto t^p$ and $H \equiv \dot{a}/a = p/t$ where $\cdot \equiv d/dt$. Note that, if $p > 1$, the expansion is superluminal. It is convenient to rewrite some of these results in terms of conformal time, τ . If $p > 1$ we have that the past is at $\tau = -\infty$ and blows up at $\tau = 0$. We then have that the scale factor and the conformal Hubble parameter is given by $a \propto (-\tau)^{\frac{p}{p-1}}$ and the conformal Hubble parameter is given by $\mathcal{H} \equiv \frac{a'}{a} = \frac{-p}{p-1} \frac{1}{\tau}$ where $' \equiv d/d\tau$.

Let us now focus on how perturbations on these background cosmologies are seeded and evolve [10]. Recall that we can expand a scalar field and space-time metric around a homogeneous background, $\phi = \phi_0 + \varphi$ and $ds^2 = a^2[(1 + 2\Phi)d\tau^2 - (1 - 2\Psi)d\mathbf{r}^2]$. The quantity of choice is the gauge invariant variable,

$$v = a(\delta\varphi + \frac{\dot{\phi}_0}{H}\Psi)$$

which can be related to the curvature perturbation, $\mathcal{R} = -v/z$ where $z = \frac{a\dot{\phi}_0}{H}$. The gauge invariant Newtonian (or “Bardeen”) potential, Φ , can be found from \mathcal{R} through $k^2\Phi = 4\pi G\dot{\phi}_0 z \mathcal{R}'$. For our choice of background cosmolo-

gies, the gauge invariant perturbation variable obeys a Bessel equation with a general solution given by:

$$v_{\mathbf{k}}(\tau) = A_{\mathbf{k}}(|\tau|)^{1/2} J_{\nu}(k|\tau|) + B_{\mathbf{k}}(|\tau|)^{1/2} Y_{\nu}(k|\tau|)$$

where $J_{\nu}(x)$ and $Y_{\nu}(x)$ are Bessel functions with $\nu = \frac{3}{2} + \frac{1}{p-1}$.

In a universe undergoing superluminal expansion there is a natural mechanism by which fluctuations can be seeded. We assume that v is promoted to a quantum operator:

$$\hat{v}(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} [v_{\mathbf{k}}(\tau) \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_{\mathbf{k}}^*(\tau) \hat{a}_{\mathbf{k}}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}}]$$

where $\hat{a}_{\mathbf{k}}$ and its conjugate are the annihilation and creation operators for the n particle state. We are interested in the two point correlation function $\langle \hat{v}(\mathbf{x}+\mathbf{r}) \hat{v}(\mathbf{x}) \rangle$, where $\langle \dots \rangle$ is a quantum expectation value depends on the state one is considering, i.e. $\langle A \rangle = \langle \zeta | \hat{A} | \zeta \rangle$. A natural choice is the ground-state or vacuum state of each mode, $|\zeta\rangle = |0\rangle$. In the past, where $(-k\tau) \rightarrow -\infty$, a given mode was well within the horizon. This allows us to uniquely define the solution (i.e. the coefficients $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$) to the mode equation to be $v_{\mathbf{k}}(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2} (-\tau)^{1/2} H_{\nu}^{(1)}(-k\tau)$ (where $H_{\nu}^{(1)}(x)$ is a Hankel function). This solution has a unique behaviour at late times (i.e. when $(-k\tau) \rightarrow 0$):

$$v_{\mathbf{k}}(\tau) \rightarrow e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma[\nu]}{\Gamma[3/2]} \frac{1}{\sqrt{2k}} (-k\tau)^{-\nu+1/2} \times [1 - \frac{(-k\tau)^2}{4(1-\nu)}]$$

\mathcal{R} can be trivially obtained from the above solution and we find that

$$\mathcal{P}_{\mathcal{R}}^0(k) = \frac{2^{2\nu-4}}{2\pi^2} \left(\frac{\Gamma[\nu]}{\Gamma[3/2]} \right)^2 \frac{(-\tau)^{-2\nu+1}}{z^2} k^{-2/(p-1)} \quad (1)$$

which goes to a constant as $p \rightarrow \infty$. We find the well known result that the scalar spectral index is given by $n_S - 1 = \frac{2}{1-p} = \frac{6(1+w)}{1+3w}$. We can see that, in the limit of $w \rightarrow -1$ we have pure scale invariance.

Throughout the above calculation we have discarded any reference to the hot origins of the universe. Yet we are starting off at high energies, when the Universe would have been strongly interacting. It would be natural to expect the imprint of these thermal initial conditions on the scalar field in some way. Indeed one would expect fluctuations in the scalar field to be thermalized through a variety of different mechanisms. The universe may have entered a scalar field dominated regime from a preceding radiation dominated regime; interactions with the hot radiation would have led the fluctuations in the scalar field to be thermal. Furthermore, the scalar field model we are considering has non-linear self-interactions through the exponential potential. Very short wave modes would play the role of a heat bath even through the period of

superluminal expansion and scalar field domination. The details of how primordial fields undergo evolution in a hot phase have been studied in great detail in [12, 13] where a number of effects where identified emerging from the non-equilibrium nature of the problem.

In what follows, we will disregard non-equilibrium effects: these will introduce small corrections and can be included in a more detail calculation. Our calculation is therefore undertaken in the setting of equilibrium statistical mechanics: the appropriate expectation value to consider is given by $\langle A \rangle = \sum_n \rho_{nn} \langle n | \hat{A} | n \rangle / (\sum_n \rho_{nn} \langle n | n \rangle)$, where $|n\rangle$ is the n -particle state (referring to a given momentum \mathbf{k}). The simplest approach is to simply posit that each mode is Boltzman weighted. Recall that this involves setting the density matrix above to $\rho_{nn} = e^{-\beta E_n}$ where E_n is the energy of a given mode with occupation number n , $\beta = 1/K_B T$, K_B is the Boltzman constant and T is the temperature. Hence we find that

$$\langle \hat{v}(\mathbf{x}+\mathbf{r}) \hat{v}(\mathbf{x}) \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} |v_{\mathbf{k}}(\tau)|^2 [2n(k) + 1] e^{i\mathbf{k}\cdot\mathbf{r}}$$

where the resulting number density (subtracting out the vacuum state) for is given by $n(k) = \frac{1}{e^{\beta E(k,\tau)} - 1}$.

The energy of the perturbation can be found from the Hamiltonian density of v . With the above solutions we have that $E(k, \tau) = \frac{\hbar\pi|\tau|}{4} B(k, \tau) |H_{\nu}(k|\tau)|$ with $B(k, \tau) = (3p^2 - p)/[\tau^2(p-1)^2] + k^2$. Note that in the short wave length limit we recover the standard for plane waves: $E(k, \tau) = \hbar k$. Also note that the energy is defined in terms of conformal quantities (derivatives are taken with regards to τ and conformal \mathbf{x}). This means that the temperature we use must also be in the same conformal frame: $T = T_{phys} a$ where T_{phys} is the physical temperature. If we assume that thermalization is maintained through a heat bath which evolves as radiation, and there is no generation of entropy during inflation, we have that T is constant. This value of T will be crucial in what follows.

We have not yet arrived at the final result. Any mechanism that keeps the scalar field in thermal equilibrium must break down as a given mode becomes larger than any causal scale, i.e. around the horizon scale. On superhorizon scales, we expect the spectrum to be frozen in—the heat bath or interactions are irrelevant. In other words $n(k)$ will be frozen at the value it has when $k|\tau| \simeq 1$. This means that the above expression is not entirely accurate and we must replace $E(k, \tau)$ with $E(k, \tau)|_{k|\tau| \simeq 1}$. We can now express our main result: the full spectrum of fluctuations, including the thermal contribution is

$$\mathcal{P}_{\mathcal{R}}^{Total}(k) = \mathcal{P}_{\mathcal{R}}^{Th}(k) + \mathcal{P}_{\mathcal{R}}^0(k) = \mathcal{P}_{\mathcal{R}}^0(k) [2(n(k, \tau)|_{k|\tau| \simeq 1} + 1)] \quad (2)$$

This result is slightly different from that in [6] (the model is not the same). It also cannot be directly compared with the results of warm inflation [13].

Let us now explore the consequences of Equation 2. As a first guess, one would expect to be in the Rayleigh-Jeans regime when the modes exit the horizon. We then have, on superhorizon scales,

$$\mathcal{P}_{\mathcal{R}}^{Th}(k) \simeq \frac{(p-1)^2}{4p^2-3p+1} \frac{(-\tau)^{-2\nu+1}}{z^2} k^{\frac{1+p}{1-p}} \frac{1}{\beta\pi^2} \quad (3)$$

If we reexpress Eq. 3 in terms of the equation of state, we have that

$$n_s - 1 = \frac{5+3w}{1+3w}$$

Close to de-Sitter we find that $n_s \simeq 0$, that is white noise: we do not get scale-invariance because the temperature is decreasing like $1/a$, breaking the deSitter invariance (this is to be contrasted with the work of [11]). Instead we find that a scale invariant spectrum arises if we assume a “phantom” regime with $w = -5/3$.

Our expression is insensitive to the details of thermalization and horizon crossing and it gives us a reasonable idea of what to expect. A useful exercise is to compare the contribution of thermal fluctuations relative to vacuum fluctuations during an inflationary period. We have that

$$\frac{\mathcal{P}_{\mathcal{R}}^{Th}}{\mathcal{P}_{\mathcal{R}}^0} \simeq \frac{(p-1)^2}{4p^2-3p+1} \frac{8}{\pi|H_\nu(1)|^2} \frac{K_B T}{\hbar k}$$

For $p \gg 1$ we find

$$\frac{\mathcal{P}_{\mathcal{R}}^{Th}}{\mathcal{P}_{\mathcal{R}}^0} \simeq 0.1 \frac{K_B T}{\hbar k}$$

In general the prediction of this model is a break in the power spectrum at pivot scale $k_p \simeq 0.1T$. For $k < k_p$ the fluctuations are predominantly thermal with spectral index $n_s^{Th} = n_s^0 - 1$, with the quantum fluctuations spectral index n_s^0 given by the usual formula. In this regime we are invariably in the Rayleigh Jeans limit. For $k > k_p$ the fluctuations are predominantly quantum, with thermal fluctuations suppressed by a factor of e^{-k/k_p} , given that we are in the Wien regime. At horizon crossing we always have $E \sim k$, so this can be replaced in the formula for $n(k)$ in either regime.

We now examine the implications of this result for two viable scenarios, where the fluctuations are predominantly quantum and thermal, respectively. If we have an inflationary scenario ($w \approx -1$) then the dominant fluctuations on observable scales should be quantum, for these are near-scale-invariant. A priori the prediction of this model is a turn over in the spectral index from $n_s \approx 1$ to $n_s = 0$ on large scales (for $k < k_p$). There is clearly no evidence for *higher* power in the lowest multipoles of the CMB so, at best k_p could be the current horizon scale k_{H_0} . This is reflected on an *upper bound* on the temperature during and before inflation or alternatively on a constraint on the ratio of the temperature before and after reheating (T_b and T_a). Recall that the conformal

temperature $T = T_{phys}a$ is only a constant if there is no entropy production, so that it does suffer a jump, from T_b to T_a at reheating. Bearing this in mind, $k_p \sim 0.1T_b$ but $k_{H_0} \sim aH_0$. Therefore $k_p < k_{H_0}$ translates into

$$\frac{T_b}{T_a} < 10 \times \frac{\hbar k_{H_0}}{K_B T_0} \simeq 10^{-28} \quad (4)$$

A marginally tighter bound can probably be obtained through the Grischuk-Zeldovich effect: superhorizon fluctuations with such a red spectrum will further boost the quadrupole [14]. We can convert our constraint into a physical temperature during inflation if we assume a specific model. For example, if the inflation ended at the GUT scale, when the energy scale is of order 10^{17} GeV and at a redshift of $z \simeq 10^{28}$, the temperature of the Universe just before reheating would have been, at most, 10^{-2} eV. This means that the Universe hits the Planck temperature more than 68 efoldings before reheating, so that there is scope for producing the observed structure of the Universe (for which 50 to 60 efoldings before reheating is enough), but, if the bound is saturated, not much more. In general the bound (4) forces the maximum number of efoldings to be

$$\mathcal{N}_{max} > \mathcal{N}_{min} + 2 \ln \frac{E_{Pl}}{E_{Inf}} - 2.3 \quad (5)$$

If we can assume that H doesn't vary by much during inflation, and if all the energy in the inflaton field is converted into radiation during reheating we can translate the bound (4) into $\mathcal{N} > 64$. Relaxing these assumptions produces a tighter bound. This seems to rule out open inflationary models.

If we have a phantom scenario with $w \approx -5/3$, the observed structure of the Universe should be thermal. The prediction is a near scale invariant spectrum breaking into $n_s = 2$ for $k > k_p$. Thus we should have $k_p > k_{S_0}$, where k_{S_0} is the smallest scale for which the primordial power spectrum is observable. The constraint is now an upper bound on how much entropy has been produced since the observed structure left the horizon; specifically:

$$\frac{T_b}{T_a} > 5 \times \frac{\hbar k_{S_0}}{K_B T_0} \simeq 10^{-22} \quad (6)$$

where we have assumed that the smallest scales that can be probed are of the order of a Kpc. In this scenario we have roughly that $a \propto 1/(-t)$, with $t < 0$ (i.e. $p = -1$), $\rho \propto a^2$, and $H^{-1} = -t$ (the horizon's physical size decreases). Also $a = -1/(2H\tau)$.

The normalization in this model is obtained from a constant of motion combining the energy in the thermal bath and that in the background field. The relevant factor in (3) is $1/(z^2\tau)$ which can be rewritten into $T^{phys}H/M_{Pl}^2$, i.e. $\rho_{\delta\phi}^{1/4}\rho_\phi^{1/2}$, or $\sim T^{phys}/|t|$. It's this important constant that must be $\sim 10^{-10}$ to match observations. Should all the energy in the “phantom” field be converted into radiation at the end of this phase we therefore get the rather undemanding bound $T_a < 10^4 M_{Pl}$ (in

combination with (6)). But by requiring that the current Hubble volume was once inside the phantom Hubble volume (in a calculation mimicking the inflationary counterpart) we find that $T^{phys}/T_{Pl} \sim 0.1$ at the start of the phantom phase (and that requires saturating bound (6)). If the thermal bath is set up at $T^{phys} \sim T_{Pl}$ the break into $n_s = 2$ should happen only an order of magnitude or so above k_{S0} . Whether this could be observed is debatable.

Note that for simplicity we have considered $w = -5/3$, but strict scale-invariance in this scenario is actually pathological as it requires it is only for $-5/3 < w < -1/3$ that the Newtonian potential Φ stays constant and has the same spectrum as \mathcal{R} on large scales. For $w \leq -5/3$ the potential diverges. However as long as the spectrum is slightly red this is not a problem and we have for the growing mode:

$$\mathcal{R} = -\frac{5+3w}{3(1+w)}\Phi \quad (7)$$

We conclude with a few comments on aspects of this model, and how they relate to other work. We stress that our system is very different from a single thermal fluid, as previously studied [4, 5, 7]. Here the unperturbed field ϕ_0 is *not* thermalized; only its fluctuations $\delta\phi$ are thermalized. The fluid ϕ_0 drives the expansion and provides the leading order energy, but no entropy. Whatever the equation of state w for ϕ_0 , the $\delta\phi$ behave like standard thermal radiation, with $w_1 = 1/3$ and supply the entirety of the entropy of the system. This feature allows us to bypass a number of thermodynamical constraints pertaining to single thermal fluids, namely the relation $\zeta = 1 + 1/w$ between the ζ exponent appearing in $\rho \propto T^\zeta$ and w . If we insist on a Stephan-Boltzman law of the form $\rho_0 \propto T^\zeta$ (where ρ_0 is the energy in ϕ_0 , and T is the temperature of $\delta\phi$) we find instead that $\zeta = 3(1+w)$. This doesn't contradict any fundamental thermodynamical constraint: the usual result merely indicates that the second order energy, contained in $\delta\phi$, should go like T^4 .

But even a two-fluid model breaks down when discussing thermal fluctuations. Indeed Maxwell's formula, $\sigma_E^2(R) = T^2 dU/dT$, which is the workhorse of much previous work [3, 4, 5], is not applicable here. The energy fluctuation is of the form $\delta\rho \sim \dot{\phi}_0 \delta\dot{\phi}$, i.e. a cross term between the unthermalized ϕ_0 and the thermalized $\delta\phi$. So the energy fluctuation of the system is, to leading order,

$\sigma_E^2(R) \propto U_0 U_1(R)$, where $U_0 = \rho_0 V$ is the average energy in ϕ_0 , and $U_1(R)$ is the average energy in $\delta\phi$ smoothed on scale R (which is $\sim T$). Unusually, we only need to know the average energy of the thermalized system to work out the leading order energy fluctuation in the overall system. These novelties conjure to bypass the general prediction $n_s = 4$, allowing for scale-invariant thermal fluctuations without appealing to any new physics.

Regarding the Gaussianity of these fluctuations it has been shown [8] that for a single thermal fluid thermal fluctuations are very approximately Gaussian in the Rayleigh-Jeans limit (but not in the Wien limit). However, just as it happens with the equivalent calculation of the variance, the calculation of the cumulants in a single thermal fluid is not applicable to our system. Instead we note that the derivation of Gaussianity usually used for linear inflation applies to any density matrix that is diagonal in the number operator, including a thermal state. We therefore expect the thermal component to be Gaussian, too, rendering the thermal scenario presented above viable. This is in contrast with non-linear inflationary couplings, that may produce a certain degree of non-Gaussianity [15].

Finally, we remind the reader that we are considering a universe that starts off in thermal equilibrium. The hallowed example is that of what has become known as new Inflation: as the Universe cools down, the scalar field settles down into a slow roll regime and it is potential energy dominated. This is not, however, a generic feature of the inflationary cosmology. One appealing alternative is a Universe that emerges through quantum tunnelling into an inflationary era [16]. Another possibility is that our local patch has entered into an inflationary regime as a result of a Planck scale fluctuation of the Inflaton [17]. The initial state for the onset inflation would not necessarily be thermal. In both of these scenarios we don't expect a thermal imprint on space time on large scales.

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